# Energy quantization and mean value inequalities for nonlinear boundary value problems

Katrin Wehrheim

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**Abstract**

We give a unified statement and proof of a class of wellknown mean value inequalities for nonnegative functions with a nonlinear bound on the Laplacian. We generalize these to domains with boundary, requiring a (possibly nonlinear) bound on the normal derivative at the boundary. These inequalities give rise to an energy quantization principle for sequences of solutions of boundary value problems that have bounded energy and whose energy densities satisfy nonlinear bounds on the Laplacian and normal derivative: One obtains local uniform bounds on the complement of finitely many points, where some minimum quantum of energy concentrates.

## 1 Energy quantization

One purpose of this note is to explain an 'energy quantization' principle that – in different forms – has successfully been applied to a variety of partial differential equations, such as minimal submanifolds, harmonic maps, pseudoholomorphic curves, and Yang-Mills connections. The common feature of these PDE's is an energy functional. (The solutions often but not necessarily are critical points thereof.) The phenomenon that we call 'energy quantization' is a consequence of a mean value inequality for the energy density. The second purpose of this note is to give a unified statement and proof of the underlying mean value inequality for the Laplace operator. In section 2 we moreover generalize this inequality to domains with boundary and inhomogeneous Neumann boundary conditions.

In this section we generally consider a PDE for maps $u : D \to T$ from a Riemannian manifold $D$ (with possibly nonempty boundary $\partial D$) to a target space $T$, e.g. another manifold, a Banach space, or a fibre bundle over $D$. The energy is given for all sufficiently regular maps $u$ in the form

$$E(u) = \int_D e(u),$$

where the integrand $e(u) : D \to [0, \infty)$ is a nonnegative energy density function. Its key property is that for a solution $u$ of the PDE, the positive definite Laplacian $\Delta e(u)$ can be bounded above in terms of $e(u)$ itself.

If this bound is linear in $e(u)$, then one immediately obtains a $\mathcal{C}^0$-control on $e(u)$ in terms of its mean values on geodesic balls. If moreover the energy $E(u_i)$ of a sequence of solutions $u_i$ is bounded, then this provides a uniform bound on the energy densities $e(u_i)$ on any compact subdomain of $D \setminus \partial D$. In many cases this leads to a compactness result, i.e. to the convergence of a subsequence of the solutions $u_i$.

For solutions of nonlinear PDE's however, the bound on $\Delta e(u)$ is usually nonlinear in $e(u)$. In that case, for polynomial nonlinearities up to some order depending on the $\dim D$, the mean value inequality only hold on geodesic balls with sufficiently small energy. In theorem 2.1 we give a precise and general statement of this wellknown fact. In theorem 2.4, we then generalize this mean value inequality to domains $D$ with boundary $\partial D$ and bounds on the outer normal derivative $\frac{\partial}{\partial \nu} e|_{\partial D}$. This allows to obtain uniform bounds on $e(u_i)$ up to the boundary for solutions $u_i$ of a PDE with appropriate boundary conditions and with bounded energy.

In the case of nonlinear bounds on the Laplacian or the normal derivative one only obtains locally uniform bounds on the complement of finitely many points: By a converse of the mean value inequality, the energy densities $e(u_i)$ can only blow up at points where some nonzero quantum of energy concentrates. In the following lemma we give a blueprint for such energy quantization results.

Here $D$ is a Riemannian manifold (possibly noncompact or with boundary), $\Delta = \mathrm{d}^*\mathrm{d}$ denotes the positive definite Laplace operator, and $\frac{\partial}{\partial \nu}$ denotes the outer normal derivative at $\partial D$. For the sake of simplicity we make a technical assumption on the metric near the boundary.

**Assumption:** A neighbourhood of $\partial D \subset D$ is locally isometric to the Euclidean half space $\mathbb{H}^n$.

For a general metric near the boundary, the mean value inequality becomes harder and more technical to prove, but theorem 2.4 should generalize in the same way as theorem 2.1, and hence this lemma should extend to general Riemannian manifolds with boundary.

**Lemma 1.1** *There exists a constant $\hbar > 0$ depending on $n = \dim D$ and some constants $a, b \geq 0$ such that the following holds: Let $e_i \in \mathcal{C}^2(D, [0, \infty))$ be a sequence of nonnegative functions such that for some constants $A_0, A_1, B_0, B_1 \geq 0$*

$$\begin{cases} \Delta e_i & \leq A_0 + A_1 e_i + a e_i^{\frac{n+2}{n}}, \\ \frac{\partial}{\partial \nu}\big|_{\partial D} e_i & \leq B_0 + B_1 e_i + b e_i^{\frac{n+1}{n}}. \end{cases}$$

*Moreover, suppose that there is a uniform bound $\displaystyle\int_D e_i \leq E < \infty$.*

*Then there exist finitely many points, $x_1, \ldots, x_N \in D$ (with $N \leq E/\hbar$) and a subsequence such that the $e_i$ are uniformly bounded on every compact subset of $D \setminus \{x_1, \ldots x_N\}$, and there is a concentration of energy $\hbar > 0$ at each $x_j$ : For every $\delta > 0$ there exists $I_{j,\delta} \in \mathbb{N}$ such that*

$$\int_{B_\delta(x_j)} e_i \geq \hbar \qquad \forall i \geq I_{j,\delta}. \tag{1}$$

**Proof:** Suppose that for some $x \in D$ there is no neighbourhood on which the $e_i$ are uniformly bounded. Then there exists a subsequence (again denoted $(e_i)$) and $D \ni z_i \to x$ such that $e_i(z_i) = R_i^n$ with $R_i \to \infty$. We can then apply the mean value inequality theorem 2.1 on the balls $B_{\delta_i}(z_i)$ of radius $\delta_i = R_i^{-\frac{1}{2}} > 0$. For sufficiently large $i \in \mathbb{N}$, these lie within appropriate coordinate charts of $D$. In case $z \in \partial D$ we use the Euclidean coordinate charts at the boundary to apply theorem 2.4, but we also denote the balls in half space by $B_{\delta_i}(z_i)$. These mean value theorems provide uniform constants $C$ and $\hbar := \mu(a,b) > 0$ such that for every $i \in \mathbb{N}$ either

$$\int_{B_{\delta_i}(z_i)} e_i > \hbar \tag{2}$$

or $\int_{B_{\delta_i}(z_i)} e_i \leq \hbar$ and hence

$$R_i^n \;=\; e(z_i) \;\leq\; CA_0\delta_i^2 + CB_0\delta_i + C\bigl(A_1^{\frac{n}{2}} + B_1^n + \delta_i^{-n}\bigr)\hbar.$$

In the latter case multiplication by $\delta_i^n = R_i^{-\frac{n}{2}}$ implies

$$R_i^{\frac{n}{2}} \;\leq\; CA_0R_i^{-\frac{n+2}{2}} + CB_0R_i^{-\frac{n+1}{2}} + C\hbar\bigl(A_1^{\frac{n}{2}}R_i^{-\frac{n}{2}} + B_1^nR_i^{-\frac{n}{2}} + 1\bigr). \tag{3}$$

As $i \to \infty$, the left hand side diverges to $\infty$, whereas the right hand side converges to $C\hbar$. Thus the alternative (2) must hold for all sufficiently large $i \in \mathbb{N}$. In particular, this implies the energy concentration (1) at $x_j = x$.

Now we can go through the same argument for any other point $x \in D$ at which the present subsequence $(e_i)$ is not locally uniformly bounded. That way we iteratively find points $x_j \in D$ such that the energy concentration (1) holds for a further subsequence $(e_i)$. Suppose this iteration yields $N > E/\hbar$ distinct points $x_1, \ldots, x_N$ (and might not even terminate after that). Then we would have a subsequence $(e_i)$ for which at least energy $\hbar > 0$ concentrates near each $x_j$. Since the points are distinct, this contradicts the energy bound $\int_D e_i \leq E$. Hence this iteration must stop after at most $\lfloor E/\hbar \rfloor$ steps, when the present subsequence $(e_i)$ is locally uniformly bounded in the complement of the finitely many points, where we found the energy concentration before. $\square$

The analogy in the compactness proofs for a variety of PDE's, including minimal submanifolds [A, CS] and harmonic maps of surfaces [SU], has already been observed and listed by Wolfson [Wo]. In all cases, the nonlinearities are exactly of the maximal order as in lemma 1.1. Here we discuss pseudoholomorphic curves and Yang-Mills connections in more detail.

For **pseudoholomorphic curves** (with a 2-dimensional domain) the energy is the $L^2$-norm of the gradient, and the estimate $\Delta e \leq C(e + e^2)$ leads to Gromov's compactness result [G, Wo]. For **pseudoholomorphic curves with Lagrangian boundary conditions**, these compactness results are also well-known. They can be proven via a specific choice of a metric for which $\frac{\partial}{\partial\nu}e = 0$ (see [F] and [MS, 4.3]). Then the energy density can be extended across the boundary by reflection and the mean value inequality for $\mathbb{R}^n$ applies. For the

metric given by the almost complex structure, the Lagrangian boundary condition only implies $\frac{\partial}{\partial\nu}e \leq C(e+e^{\frac{3}{2}})$, which fits nicely into our energy quantization principle.

For **Yang-Mills connections** on 4-manifolds the energy is the $L^2$-norm of the curvature. The bound $\Delta e \leq C(e+e^{\frac{3}{2}})$ was used by Uhlenbeck [U] to prove a removable singularity result, which leads to Donaldson's compactification of the moduli space of anti-self-dual instantons [D]. For a proof of the energy quantization as in lemma 1.1 see also [We, Thm.2.1].

For **anti-self-dual instantons with Lagrangian boundary conditions** on $\mathbb{H}^2 \times \Sigma$, an argument along the lines of lemma 1.1 is used in [We, Thm.1.2]. There the energy density is given by the slicewise $L^2$-norm of the curvature, $e(A) = \int_\Sigma |F_A|^2$. The Lagrangian boundary condition (which has global nature along the Riemann surface $\Sigma$) provides $\frac{\partial}{\partial\nu}e \leq C(e+e^{\frac{3}{2}})$, but one only has a linear bound $\Delta e \leq ge$ with a function $g$ that cannot be bounded in terms of $e$ or a constant. The argument using (3) however allows for a mild blowup of $A_1$, and the according estimate $|g| \leq CR_i^2$ can be established. This result does not follow from the standard rescaling methods for Yang-Mills connections.

# 2 Mean value inequalities

In this section we state and prove the mean value inequalities that were desribed in section 1 and that the energy quantization principle is based on.

The subsequent theorem is wellknown and proofs in an exhausting collection of cases can be found in the literature, e.g. [S, U]. Our aim here is to give a unified statement and proof. In theorem 2.4 below we moreover generalize this result to domains with boundary.

We denote by $B_{r_0}(x_0) \subset \mathbb{R}^n$ the open geodesic ball of radius $r_0$ centered at $x_0 \in \mathbb{R}^n$ and with respect to the present metric. The Laplace operator $\Delta = \mathrm{d}^*\mathrm{d}$ and integration will also be using the metric given in the context. The Euclidean metric on $\mathbb{R}^n$ is denoted by its matrix $\mathbb{1}$.

**Theorem 2.1** *For every $n \in \mathbb{N}$ there exist constants $C$, $\mu > 0$, and $\delta > 0$ such that the following holds for all metrics $g$ on $\mathbb{R}^n$ such that $\|g - \mathbb{1}\|_{W^{1,\infty}} \leq \delta$.*

*Let $B_r(0) \subset \mathbb{R}^n$ be the geodesic ball of radius $0 < r \leq 1$. Suppose that $e \in \mathcal{C}^2(B_r(0), [0,\infty))$ satisfies for some $A_0, A_1, a \geq 0$*

$$\Delta e \leq A_0 + A_1 e + a\, e^{\frac{n+2}{n}} \qquad \text{and} \qquad \int_{B_r(0)} e \leq \mu a^{-\frac{n}{2}}.$$

*Then*

$$e(0) \leq CA_0 r^2 + C\big(A_1^{\frac{n}{2}} + r^{-n}\big) \int_{B_r(0)} e.$$

**Remark 2.2** *By using local geodesic coordinates the above theorem also implies a mean value inequality on closed Riemannian manifolds with uniform constants $C, \mu > 0$, and for all geodesic balls of radius less than a uniform constant.*

The special case $A_0 = A_1 = a = 0$ of theorem 2.1 and the starting point for the proof is Morrey's [M] mean value inequality for subharmonic functions. A proof of the version below can be found in e.g. [LS, Thm.2.1]. For the Euclidean metric $g = \mathbb{1}$ we give an elementary proof in lemma 2.5 below.

**Proposition 2.3** *For every $n \in \mathbb{N}$ there exist constants $C$ and $\delta > 0$ such that the following holds for all $0 < r \leq 1$ and all metrics $g$ on $\mathbb{R}^n$ with $\|g - \mathbb{1}\|_{W^{1,\infty}} \leq \delta$: If $e \in \mathcal{C}^2(B_r(0), [0, \infty))$ satisfies $\Delta e \leq 0$ then*

$$e(0) \leq C r^{-n} \int_{B_r(0)} e.$$

**Proof of theorem 2.1:**
This proof is based on the Heinz trick, which is to consider the maximum $\bar{c}$ of the function $f$ below. This allows to replace the bound on the Laplacian by a constant depending on $\bar{c}$. One then obtains the result from the mean value inequality for subharmonic functions and a number of rearrangements in different cases.

Consider the function $f(\rho) = (1-\rho)^n \sup_{B_{\rho r}(0)} e$ for $\rho \in [0, 1]$. It attains its maximum at some $\bar{\rho} < 1$. Let $\bar{c} = \sup_{B_{\bar{\rho} r}(0)} e = e(\bar{x})$ and $\varepsilon = \frac{1}{2}(1-\bar{\rho}) < \frac{1}{2}$, then

$$e(0) \;=\; f(0) \;\leq\; f(\bar{\rho}) \;=\; 2^n \varepsilon^n \bar{c}.$$

Moreover, we have for all $x \in B_{\varepsilon r}(\bar{x}) \subset B_r(y)$

$$e(x) \;\leq \sup_{B_{(\bar{\rho}+\varepsilon)r}(0)} e \;=\; \bigl(1 - \bar{\rho} - \varepsilon\bigr)^{-n} f(\bar{\rho} + \varepsilon) \;\leq\; 2^n (1 - \bar{\rho})^{-n} f(\bar{\rho}) \;=\; 2^n \bar{c},$$

and hence $\Delta e \leq A_0 + 2^n A_1 \bar{c} + 2^{n+2} a \bar{c}^{\frac{n+2}{n}}$. Now define the function

$$v(x) := e(x) + \tfrac{1}{n}\bigl(A_0 + 2^n \bar{c}\bigl(A_1 + 4a\bar{c}^{\frac{2}{n}}\bigr)\bigr)|x - \bar{x}|^2$$

with the Euclidean norm $|x - \bar{x}|$. It is nonnegative and subharmonic on $B_{\varepsilon r}(\bar{x})$ if the metric is sufficiently $\mathcal{C}^1$-close to $\mathbb{1}$. This is since $\Delta_{\mathbb{1}} |x - \bar{x}|^2 = 2n$ for the Euclidean metric and $|x - \bar{x}| \leq \varepsilon r \leq 1$ is bounded, so $\Delta |x - \bar{x}| \geq n$ whenever $\|g - \mathbb{1}\|_{W^{1,\infty}} \leq \delta$ is sufficiently small. The control of the metric also ensures that the integral $\int_{B_{\rho r}(\bar{x})} |x - \bar{x}|^2$ is bounded by the following integral over the Euclidean ball $B^{\mathbb{1}}_{2\rho r}(\bar{x})$

$$2 \int_{B^{\mathbb{1}}_{2\rho r}(\bar{x})} |x - \bar{x}|^2 \;=\; 2 \int_0^{2\rho r} t^{n+1} \mathrm{Vol}\, S^{n-1} \,\mathrm{d}t \;=\; C_n (\rho r)^{n+2}.$$

So we obtain from proposition 2.3 with a uniform constant $C$ for all $0 < \rho \leq \varepsilon$

$$\bar{c} \;=\; v(\bar{x}) \;\leq\; C(\rho r)^{-n} \int_{B_{\rho r}(\bar{x})} v \;\leq\; C\bigl(A_0 + \bar{c}\bigl(A_1 + 4a\bar{c}^{\frac{2}{n}}\bigr)\bigr)(\rho r)^2 + C(\rho r)^{-n} \int_{B_{\rho r}(\bar{x})} e.$$

If $CA_0(\varepsilon r)^2 \leq \frac{1}{2}\bar{c}$, then we can drop $A_0$ from this inequality, just changing the constant $C$. Otherwise $e(0) \leq \bar{c} \leq C A_0 r^2$ already proves the assertion.

Next, if $C(A_1+4a\bar{c}^{\frac{2}{n}})(\rho r)^2 \leq \frac{1}{2}$, then this implies $\bar{c} \leq 2C(\rho r)^{-n} \int_{B_r(0)} e$. So if $C(A_1+4a\bar{c}^{\frac{2}{n}})(\varepsilon r)^2 \leq \frac{1}{2}$ then $\rho = \varepsilon$ proves the assertion,

$$e(0) \;\leq\; 2^n \varepsilon^n \bar{c} \;\leq\; 2^{n+1} C r^{-n} \int_{B_r(0)} e.$$

Otherwise we can choose $0 < \rho < \varepsilon$ such that $\rho r = \big(2C(A_1+4a\bar{c}^{\frac{2}{n}})\big)^{-\frac{1}{2}}$ to obtain with a new uniform constant $C$

$$e(0) \;\leq\; \bar{c} \;\leq\; C\big(A_1+4a\bar{c}^{\frac{2}{n}}\big)^{\frac{n}{2}} \int_{B_{\rho r}(\bar{x})} e.$$

Again we have to distinguish two cases: Firstly, if $4a\bar{c}^{\frac{2}{n}} \leq A_1$ then this yields

$$e(0) \;\leq\; C(2A_1)^{\frac{n}{2}} \int_{B_{\rho r}(\bar{x})} e.$$

Secondly, if $A_1 < 4a\bar{c}^{\frac{2}{n}}$ then $\bar{c} < \bar{c}\, C(8a)^{\frac{n}{2}} \int_{B_{\rho r}(\bar{x})} e$ and thus with some $\mu > 0$

$$\int_{B_r(0)} e \;>\; \mu a^{-\frac{n}{2}}.$$

So altogether we either have the above or with some uniform constant $C$

$$e(0) \;\leq\; C A_0 r^2 + C\big(A_1^{\frac{n}{2}} + r^{-n}\big) \int_{B_r(0)} e.$$

□

Next we generalize the mean value inequality theorem 2.1 to the half space

$$\mathbb{H}^n = \{(x_0,\bar{x}) \,\big|\, x_0 \in [0,\infty), \bar{x} \in \mathbb{R}^{n-1}\}.$$

In order to generalize the mean value inequality to manifolds with boundary we would have to consider general metrics on $\mathbb{H}^n$. This would however disturb the elementary geometric proof, so we restrict this exposition to the case of the Euclidean metric. In that case the outer normal derivative $\frac{\partial}{\partial \nu}|_{\partial\mathbb{H}^n}$ is just $-\frac{\partial}{\partial x_0}|_{x_0=0}$. We moreover denote the intersection of a Euclidean ball with the half space by

$$D_{r_0}(x_0) := B_{r_0}(x_0) \cap \mathbb{H}^n.$$

**Theorem 2.4** *For every $n \geq 2$ there exists a constant $C$ and for all $a, b \geq 0$ there exists $\mu(a,b) > 0$ such that the following holds:*

*Let $D_r(y) \subset \mathbb{H}^n$ be the Euclidean $n$-ball of radius $r > 0$ and center $y \in \mathbb{H}^n$ intersected with the half space. Suppose that $e \in \mathcal{C}^2(D_r(y),[0,\infty))$ satisfies for some $A_0, A_1, B_0, B_1 \geq 0$*

$$\begin{cases} \Delta e & \leq A_0 + A_1 e + a e^{\frac{n+2}{n}}, \\ \frac{\partial}{\partial \nu}\big|_{\partial\mathbb{H}^n} e & \leq B_0 + B_1 e + b e^{\frac{n+1}{n}}, \end{cases} \qquad \text{and} \qquad \int_{D_r(y)} e \leq \mu(a,b).$$

*Then*

$$e(y) \leq C A_0 r^2 + C B_0 r + C\big(A_1^{\frac{n}{2}} + B_1^n + r^{-n}\big) \int_{D_r(y)} e.$$

We will prove this theorem in three steps. The first step is the generalization of proposition 2.3 to domains with boundary and subharmonic functions in the sense of the weak Neumann equation: A distribution $e$ on a manifold $M$ is called subharmonic if for all $\psi \in \mathcal{C}^\infty(M,[0,\infty))$ with $\frac{\partial\psi}{\partial\nu}|_{\partial M} = 0$

$$0 \;\geq\; \int_M e\,\Delta\psi \;=\; \int_M \psi\,\Delta e + \int_{\partial M} \psi\,\tfrac{\partial e}{\partial \nu}.$$

For $e \in \mathcal{C}^2(M)$ the equality above holds and implies that $\Delta e \leq 0$ and $\frac{\partial e}{\partial\nu}|_{\partial M} \leq 0$.

**Lemma 2.5** *There exists a constant $C$ such that the following holds for all $R > 0$ and $y \in \mathbb{H}^n$: Suppose that $e \in \mathcal{C}^2(D_r(y),[0,\infty))$ satisfies*

$$\begin{cases} \Delta e & \leq 0, \\ \frac{\partial}{\partial\nu}\big|_{\partial\mathbb{H}^n} e & \leq 0. \end{cases}$$

*Then*

$$e(y) \leq C r^{-n} \int_{D_r(y)} e.$$

**Proof:** We write $\mathbb{H}^n = \{(x_0,\bar{x}) \,\big|\, x_0 \in [0,\infty), \bar{x} \in \mathbb{R}^{n-1}\}$ and use spherical coordinates $(x_0,\bar{x}) = (y_0 + r\cos\phi\,,\, \bar{y} + r\sin\phi\cdot z) =: (r,\phi,z)$ with $r \in [0,\infty)$, $\phi \in [0,\pi]$, and $z \in S^{n-2} \subset \mathbb{R}^n$. (For $n = 2$ this notation means $S^0 = \{-1,1\}$, and integration $\int_{S^0} \ldots \,\mathrm{dvol}_{S^0}$ will denote summation of the values at these two points.) Now the boundary of $D_r(y)$ has two parts,

$$\begin{aligned} Z_r &:= \partial D_r(y) \cap \partial\mathbb{H}^n \;=\; \big\{(0\,,\,\bar{x}) \,\big|\, |\bar{x}-\bar{y}|^2 \leq r^2 - {y_0}^2\big\}, \\ \Gamma_r &:= \partial B_r(y) \cap \mathbb{H}^n \;=\; \big\{(r,\phi,z) \,\big|\, \phi \in [0,\phi_0(r)], z \in S^{n-2}\big\}. \end{aligned}$$

Here we use $\phi_0(r) := \arccos(-y_0/r)$. For $y_0 > r$ we set $\phi_0(r) := \pi$, so $\Gamma_r$ is the entire sphere and the set $Z_r$ is empty. With this we calculate for all $r > 0$

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}r}\left(r^{-n+1}\int_{\Gamma_r} e\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}r}\left(r^{-n+1}\int_0^{\phi_0(r)}\int_{S^{n-2}} e(r,\phi,z)\,(r\sin\phi)^{n-2}\,\mathrm{dvol}_{S^{n-2}}\,r\,\mathrm{d}\phi\right) \\ &= \int_0^{\phi_0(r)}\int_{S^{n-2}} \partial_r e(r,\phi,z)\,(\sin\phi)^{n-2}\,\mathrm{dvol}_{S^{n-2}}\,\mathrm{d}\phi \\ &\quad + \frac{\partial\phi_0}{\partial r}\int_{S^{n-2}} e(r,\phi_0(r),z)\,(\sin\phi_0(r))^{n-2}\,\mathrm{dvol}_{S^{n-2}}. \end{aligned} \tag{4}$$

Note that $\phi_0(r)$ is constant for $y_0 = 0$ as well as for $r \leq y_0$. So firstly in case $y_0 > 0$ we have for all $0 < r \leq y_0$

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^{-n+1}\int_{\Gamma_r} e\right) \;=\; r^{-n+1}\int_{\partial D_r(y)} \tfrac{\partial}{\partial\nu} e \;=\; -r^{-n+1}\int_{D_r(y)} \Delta e \;\geq\; 0. \tag{5}$$

In that case we moreover have

$$\lim_{r\to 0}\left(r^{-n+1}\int_{\Gamma_r} e\right) \;=\; \operatorname{Vol} S^{n-1}\, e(y), \tag{6}$$

so integrating $\int_0^{\frac{R}{2}} r^{n-1} \ldots \,\mathrm{d}r$ proves the lemma for all $R \le 2y_0$,

$$\tfrac{1}{n}2^{-n}R^n \operatorname{Vol} S^{n-1}\, e(y) \;\le\; \int_0^{\frac{R}{2}} \int_{\Gamma_r} e \,\mathrm{d}r \;\le\; \int_{D_R(y)} e.$$

Next, in case $y_0 = 0$ we have for all $r > 0$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^{-n+1}\int_{\Gamma_r} e\right) &= r^{-n+1}\int_{\Gamma_r} \tfrac{\partial}{\partial\nu} e \\ &= -r^{-n+1}\int_{D_r(y)} \Delta e \;-\; r^{-n+1}\int_{Z_r} \tfrac{\partial}{\partial\nu} e \;\ge\; 0.\end{aligned}$$

Since $\lim_{r\to 0}\left(r^{-n+1}\int_{\Gamma_r} e\right) = \frac{1}{2}\operatorname{Vol} S^{n-1}\, e(y)$, integration over $0 < r \le R$ then proves the lemma for $y_0 = 0$ and all $R > 0$,

$$\tfrac{1}{2n}R^n \operatorname{Vol} S^{n-1}\, e(y) \;\le\; \int_0^R \int_{\Gamma_r} e \,\mathrm{d}r \;=\; \int_{D_R(y)} e.$$

Finally, in case $R > 2y_0 > 0$ we obtain from (4) for all $r > y_0$

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^{-n+1}\int_{\Gamma_r} e\right) \quad\ge\; \frac{-y_0}{r\sqrt{r^2 - y_0^2}} \int_{S^{n-2}} e(r, \phi_0(r), z)\, (\sin\phi_0(r))^{n-2}\, \mathrm{dvol}_{S^{n-2}}.$$

Now we can use (6), (5), and integrate the above to obtain for all $y_0 < r \le \frac{1}{2}R$

$$\begin{aligned}\operatorname{Vol} S^{n-1}\, e(y) \le{}& r^{-n+1}\int_{\Gamma_r} e \\ &+ \int_{y_0}^{r} y_0\, \rho^{1-n}(\rho^2 - y_0^2)^{\frac{n-3}{2}} \int_{S^{n-2}} e(\rho, \phi_0(\rho), z)\, \mathrm{dvol}_{S^{n-2}}\, \mathrm{d}\rho.\end{aligned}$$

Since $(\rho, \phi_0(\rho), z) \in \partial\mathbb{H}^n$, we already know that

$$e(\rho, \phi_0(\rho), z) \;\le\; \frac{2n}{\operatorname{Vol} S^{n-1} (\frac{R}{2})^n} \int_{D_{\frac{R}{2}}(\rho, \phi_0(\rho), z)} e \;\le\; \frac{2^{n+1} n}{\operatorname{Vol} S^{n-1} R^n} \int_{D_R(y)} e.$$

With this we find that for all $0 < y_0 < r \le \frac{1}{2}R$

$$\begin{aligned}\operatorname{Vol} S^{n-1}\, e(y) \le{}& r^{-n+1}\int_{\Gamma_r} e \\ &+ \frac{2^{n+1} n \operatorname{Vol} S^{n-2}}{R^n \operatorname{Vol} S^{n-1}} \int_1^{r y_0^{-1}} t^{-2}\big(1 - t^{-2}\big)^{\frac{n-3}{2}}\, \mathrm{d}t \int_{D_R(y)} e \\ \le{}& r^{-n+1}\int_{\Gamma_r} e \;+\; C_n R^{-n} \int_{D_R(y)} e. \end{aligned} \tag{7}$$

Here we have introduced a constant $C_n$ that only depends on $n \geq 2$, in particular on the value of the integral in $t$: For $n = 2$ we calculate it explicitly,

$$\int_1^{ry_0^{-1}} t^{-2}\big(1 - t^{-2}\big)^{-\frac{1}{2}}\,\mathrm{d}t \;=\; \Big[\arccos(t^{-1})\Big]_1^{ry_0^{-1}} \;=\; \arccos\big(\tfrac{r}{y_0}\big) \;<\; \tfrac{\pi}{2}.$$

For $n \geq 3$ we have

$$\int_1^{ry_0^{-1}} t^{-2}\big(1 - t^{-2}\big)^{\frac{n-3}{2}}\,\mathrm{d}t \;\leq\; \int_1^{ry_0^{-1}} t^{-2}\mathrm{d}t \;=\; 1 - \tfrac{r}{y_0} \;<\; 1.$$

Now from (5) we know that (7) also holds for $0 < r \leq y_0$ (with $C_n = 0$), so integrating $\int_0^{\frac{R}{2}} r^{n-1} \dots \mathrm{d}r$ proves the lemma in this last case,

$$\tfrac{1}{n}\big(\tfrac{R}{2}\big)^n \mathrm{Vol}\, S^{n-1}\, e(y) \;\leq\; \int_0^{\frac{R}{2}}\int_{\Gamma_r} e\,\mathrm{d}r \;+\; \tfrac{1}{n}\big(\tfrac{R}{2}\big)^n C_n R^{-n}\int_{D_R(y)} e \;\leq\; C\int_{D_R(y)} e.$$

$\square$

**Proof of theorem 2.4:**
With lemma 2.5 in hand, the second step of the proof is to assume constant positive bounds, $\Delta e \leq A$ and $\frac{\partial}{\partial\nu}\big|_{\partial\mathbb{H}^n} e \leq B$ and find a constant $C$ such that for all $r > 0$ and $y \in \mathbb{H}^n$

$$e(y) \;\leq\; Cr^{-n}\int_{D_r(y)} e \;+\; CAr^2 + CBr. \tag{8}$$

That is, we first prove the theorem with $A_1 = B_1 = a = b = 0$. To do this consider the function

$$v(x) := e(x) + \tfrac{1}{2n}A|x-y|^2 + (B + \tfrac{1}{n}A\, y_0)x_0.$$

It is positive and satisfies $\Delta v \leq 0$ and $\frac{\partial}{\partial\nu}\big|_{\partial\mathbb{H}^n} v \leq 0$, so lemma 2.5 implies that

$$e(y) \;=\; v(y) - (B + \tfrac{1}{n}A\, y_0)y_0 \;\leq\; v(y) \;\leq\; Cr^{-n}\int_{D_r(y)} v. \tag{9}$$

In case $r \leq y_0$ we just use $v(x) = e(x) + \frac{1}{2n}A|x-y|^2$, then the same holds, and moreover

$$\int_{D_r(y)} v \;=\; \int_{D_r(y)} e \;+\; \tfrac{1}{2n}A\int_0^r t^{n+1}\mathrm{Vol}\, S^{n-1}\mathrm{d}t \;=\; \int_{D_r(y)} e \;+\; \tfrac{\mathrm{Vol}\, S^{n-1}}{2n(n+2)}Ar^{n+2}.$$

In case $r > y_0$ we have (using $x_0 \leq 2r$ on $B_r(y)$)

$$\begin{aligned}\int_{D_r(y)} v &\leq \int_{D_r(y)} e \;+\; \tfrac{1}{2n}A\int_0^r t^{n+1}\mathrm{Vol}\, S^{n-1}\mathrm{d}t \;+\; (B + \tfrac{1}{n}A\, y_0)\int_{D_r(y)} x_0 \\ &\leq \int_{D_r(y)} e \;+\; \tfrac{\mathrm{Vol}\, S^{n-1}}{2n(n+2)}Ar^{n+2} \;+\; (B + \tfrac{1}{n}A\, r)\tfrac{2}{n}\mathrm{Vol}\, S^{n-1} r^{n+1}.\end{aligned}$$

In any case, putting this into (9) proves (8).

Finally, to prove the theorem we consider – analogous to the proof of theorem 2.1 – the function $f(\rho)=(1-\rho)^n\sup_{D_{\rho r}(y)}e$ defined for $\rho\in[0,1]$. It attains its maximum at some $\bar\rho<1$. We denote $\bar c=\sup_{D_{\bar\rho r}(y)}e=e(\bar x)$ and $\varepsilon=\frac12(1-\bar\rho)$, then $e(y)\le 2^n\varepsilon^n\bar c$ and $e(x)\le 2^n\bar c$ for all $x\in D_{\varepsilon r}(\bar x)$. Thus on $D_{\varepsilon r}(\bar x)\subset D_r(y)$ we have $\Delta e\ \le\ A_0+2^n\bar c(A_1+4a\bar c^{\frac2n})$ and $\frac{\partial}{\partial\nu}\big|_{\partial\mathbb{H}^n}e\ \le\ B_1+2^n\bar c(B_1+2b\bar c^{\frac1n})$. Putting this into (8) yields for all $0<\rho\le\varepsilon$

$$\begin{aligned}\bar c\ =\ e(\bar x)\ \le\ C(\rho r)^{-n}\textstyle\int_{D_{\rho r}(\bar x)}e\ &+C\big(A_0+2^n\bar c(A_1+4a\bar c^{\frac2n})\big)(\rho r)^2\\ &+C\big(B_0+2^n\bar c(B_1+2b\bar c^{\frac1n})\big)\rho r.\end{aligned}\tag{10}$$

To deduce the claimed mean value inequality from this, we have to go through a number of different cases. Firstly, if $CA_0(\varepsilon r)^2+CB_0\varepsilon r\ge\frac12\bar c$, then since $\varepsilon\le\frac12$

$$e(y)\ \le\ \bar c\ \le\ CA_0r^2+CB_0r,$$

which proves the theorem. Otherwise (10) continues to hold with $A_0$ and $B_0$ dropped (and another constant). Next, let $0<\varepsilon'<\varepsilon$ be the solution of $A_1(\varepsilon'r)^2+B_1\varepsilon'r=2^{-n-1}C^{-1}$ or in case $A_1(\varepsilon r)^2+B_1\varepsilon r\le 2^{-n-1}C^{-1}$ let $\varepsilon=\varepsilon'$. Then we can rearrange (10) to obtain for all $0<\rho\le\varepsilon'$ and yet another constant

$$\textstyle\int_{D_r(y)}e\ \ge\ \int_{D_{\rho r}(\bar x)}e\ \ge\ \bar c(\rho r)^n\big(C^{-1}-a\bar c^{\frac2n}(\rho r)^2-b\bar c^{\frac1n}\rho r\big).\tag{11}$$

Now if $a,b\ne0$ let $\eta(a,b)>0$ be the solution of

$$a\eta^2+b\eta=\tfrac12C^{-1}.$$

If $\bar c^{\frac1n}\rho r=\eta(a,b)$ for some $0<\rho\le\varepsilon'$, then the theorem holds by the following definition of $\mu(a,b)>0$,

$$\textstyle\int_{D_r(y)}e\ \ge\ \frac{1}{2C}\eta(a,b)^n\ =:\ \mu(a,b).$$

Otherwise we must have $\bar c^{\frac1n}\varepsilon'r<\eta(a,b)$, so (11) with $\rho=\varepsilon'$ gives

$$\bar c\ \le\ 2C(\varepsilon'r)^{-n}\textstyle\int_{D_r(y)}e.\tag{12}$$

In the special case $a=b=0$ we get the same directly from (11). In case $\varepsilon'=\varepsilon$ this proves the theorem since $e(y)\le 2^n\varepsilon^n\bar c$. Otherwise $\varepsilon'<\varepsilon$ satisfies with another constant

$$\begin{aligned}2C^{-1}&=A_1(\varepsilon'r)^2+B_1\varepsilon'r+C^{-1}\\&=\big(\sqrt{A_1}\varepsilon'r+C^{-\frac12}\big)^2+\big(B_1-2C^{-\frac12}\sqrt{A_1}\big)\varepsilon'r\\&=\big(\tfrac12C^{\frac12}B_1\varepsilon'r+C^{-\frac12}\big)^2+\big(A_1-\tfrac{C}{4}B_1^2\big)(\varepsilon'r)^2.\end{aligned}$$

From this one sees that either $B_1\le 2C^{-\frac12}\sqrt{A_1}$ and $\varepsilon'r\ge(\sqrt2-1)C^{-\frac12}A_1^{-\frac12}$ from the second line, or $A_1\le\frac C4B_1^2$ and $\varepsilon'r\ge2(\sqrt2-1)C^{-1}B_1^{-1}$ from the third line. Putting this into (12) we finally obtain in this last case

$$e(y)\ \le\ \bar c\ \le\ C\big(A_1^{\frac n2}+B_1^n\big)\textstyle\int_{D_r(y)}e.\qquad\square$$

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